Scattering Solutions of the Biedenharn Symmetric Dirac-Coulomb Hamiltonian*

D. M. FRADKIN
Institute for Atomic Research and Department of Physics, Iowa State University, Ames, Iowa
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Scattering solutions are obtained for the Biedenharn Hamiltonian which possesses R_4 symmetry. These are alternatively written in the form of an operator times a plane-wave spinor, and in this form are briefly compared with the exact Dirac solutions and the Sommerfield-Maue solutions.

I. INTRODUCTION

RECENTLY, Biedenharn[1] developed a "symmetric Dirac-Coulomb Hamiltonian" which has the symmetry of the four-dimensional rotation group R_4 and differs from the usual Dirac-Coulomb Hamiltonian by order $(\alpha Z)^2$. The R_4 symmetry of the corresponding nonrelativistic (Schrödinger-Coulomb) problem permits separation of the wave equation in parabolic coordinates which in turn provides continuum scattering solutions having a simple form.[2] However, the Dirac-Coulomb Hamiltonian does not possess this symmetry, is separable only in spherical coordinates, and the continuum scattering solutions are given only in terms of an infinite series of angular-momentum eigenfunctions. For purposes of calculation, various approximations of the exact continuum scattering solution must be made.

The Dirac-Coulomb Hamiltonian is simpler than the Biedenharn symmetric Hamiltonian, but the angular-momentum eigenfunctions of the former are more complicated than those of the latter. Thus, in terms of approximation physics, the availability of the Biedenharn symmetric Hamiltonian allows one to move the approximation from the eigenfunction to the Hamiltonian, where it may be discussed more fluently by perturbation methods.

The Biedenharn symmetric Hamiltonian has recently[3] been discussed in some detail, and the angular-momentum eigenfunctions have been displayed. The purpose of this note is to explicitly exhibit the scattering solutions for this Hamiltonian. These solutions are also cast into the form of an operator acting on a plane wave spinor of arbitrary polarization. This form is particularly convenient for calculating matrix elements, and allows a simple treatment of elastic scattering.[4] Also, a direct comparison with the corresponding form for the Dirac-Coulomb Hamiltonian[4–6] shows the simplifications that occur for the continuum scattering solutions of the symmetric Hamiltonian.

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2. THE SCATTERING SOLUTIONS

In the system of units for which $\hbar=m=c=1$, the Biedenharn symmetric Hamiltonian is[7]

$$H_B=-i\rho_1\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}+\rho_3-\alpha Z/r+\rho_2\boldsymbol{\sigma}\cdot\hat{r}K/r \times\{[1+(\alpha Z/K)^2]^{\frac{1}{2}}-1\}. \quad (1)$$

Here, the Dirac operator K is defined by

$$K=\rho_3(\boldsymbol{\sigma}\cdot\mathbf{L}+1), \quad (2)$$

and the Coulomb potential is attractive for $Z>0$. By the usual methods[7] we find that the general continuum scattering solution that asymptotically behaves like a "plane wave" of energy E and momentum p traveling in the $\hat{p}$ direction plus an outgoing spherical wave, is given by[8]

$$\psi=4\pi\left(\frac{\pi}{2Ep}\right)^{\frac{1}{2}}\sum_{m,k,\mu}c_m e^{i\Delta_k}C[l(k),\tfrac{1}{2},j;\mu-m,m] \times Y_{l(k)}{}^{*\mu-m}(\hat{p})\psi_k{}^{\mu}(\mathbf{r},E). \quad (3)$$

Here, $l(k)=|k|+\frac{1}{2}(s_k-1)$, $j=|k|-\frac{1}{2}$, $k=\pm1,\pm2,\cdots$, $s_k=\pm1$ for $k\gtrless0$, μ is a half-integer, and the summation extends over all k, $m=\pm\frac{1}{2}$, and μ such that $|\mu-m|\leq l(k)$. The phase factor Δ_k is given by

$$\Delta_k=\eta-\arg\Gamma(|k|+i\nu)+\tfrac{1}{2}\pi(|k|+s_k), \quad (4)$$
$$e^{2i\eta}=e^{-i\pi}[e^{i\pi(1-s_k)/2}(k^2+\lambda^2)^{\frac{1}{2}}-i\lambda/p]/(|k|+i\nu),$$

where the interaction strength λ and the Born parameter ν are defined by

$$\lambda=\alpha Z,\quad \nu=\alpha ZE/p. \quad (5)$$

The constants c_m are limited only by the condition $\sum_m c_m{}^*c_m=1$.

In Eq. (3), the angular-momentum eigenfunction $\psi_k{}^{\mu}(\mathbf{r},E)$ satisfies

$$(H_B-E)\psi_k{}^{\mu}(\mathbf{r},E)=0,\quad (K+k)\psi_k{}^{\mu}(\mathbf{r},E)=0, \quad (6)$$
$$\int\psi_k{}^{\mu\dagger}(\mathbf{r},E)\psi_{k'}{}^{\mu'}(\mathbf{r},E')d\mathbf{r}=\delta_{\mu,\mu'}\delta_{k,k'}\delta(E-E').$$

It is explicitly given by

$$\psi_k{}^{\mu}(\mathbf{r},E)=\begin{pmatrix} g_k(E,r)\chi_k{}^{\mu}(\hat{r}) \\ if_k(E,r)\chi_{-k}{}^{\mu}(\hat{r})\end{pmatrix} \quad (7)$$

[7] In general, we follow the notation employed by M. E. Rose, *Relativistic Electron Theory* (John Wiley & Sons, Inc., New York, 1961).

[8] The normalization of Eq. (3) is chosen so that the coefficient of the plane wave asymptotically approaches one.

in which

$$\chi_k{}^\mu(\hat{r}) = \sum_{\tau=\pm\frac{1}{2}} C[l(k),\tfrac{1}{2},j;\mu-\tau,\tau]Y_{l(k)}{}^{\mu-\tau}(\hat{r})\chi^\tau,$$

$$g_k(E,r) = [p(E+1)/\pi]^{\frac{1}{2}}\{\ \}_+,$$

$$f_k(E,r) = i[p(E-1)/\pi]^{\frac{1}{2}}\{\ \}_-,$$

$$\{\ \}_\pm = [e^{\nu\pi/2}|\Gamma(|k|+i\nu)|\,|x|^{|k|-1}/\Gamma(2|k|+1)] \times\{(|k|+i\nu)e^{i\eta}e^{-x/2} \times{}_1F_1(|k|+1+i\nu,2|k|+1,x)\pm\text{c.c.}\},$$

$$x=-2ipr.$$

By means of the methods described in Ref. 6, the scattering solution given by Eq. (3) may be written in the Johnson-Deck form

$$\psi=\{N+i\lambda M\gamma_5\boldsymbol{\sigma}\cdot(\hat{p}-\hat{r})+L[\boldsymbol{\sigma}\cdot(\hat{p}-\hat{r})](\boldsymbol{\sigma}\cdot\hat{p})\}U(\hat{p}), \quad (8)$$

and consequently all the results[4] pertaining to this form may be used. Here, $U(\hat{p})$ is a plane wave spinor of arbitrary polarization. The functions N, M, and L are given by

$$N=2\sum_{k=1}^{\infty}(-1)^k x^{k-1}e^{x/2}e^{\nu\pi/2} \times[\Gamma(k-i\nu)/\Gamma(2k+1)](k^2+\lambda^2)^{\frac{1}{2}} \times{}_1F_1(k-i\nu,2k+1,x)[P_{k-1}'(\hat{p}\cdot\hat{r})-P_k'(\hat{p}\cdot\hat{r})], \quad (9)$$

$$M=-\tfrac{1}{2}\Gamma(1-i\nu)e^{\nu\pi/2}e^{i\mathbf{p}\cdot\mathbf{r}}{}_1F_1[1+i\nu,2,i(pr-\mathbf{p}\cdot\mathbf{r})], \quad (10)$$

$$L=\tfrac{1}{2}(N_{\lambda=0}-N). \quad (11)$$

For $\lambda=0$, the series for N can be summed to yield

$$N_{\lambda=0}=\Gamma(1-i\nu)e^{\nu\pi/2}e^{i\mathbf{p}\cdot\mathbf{r}}{}_1F_1[i\nu,1,i(pr-\mathbf{p}\cdot\mathbf{r})]. \quad (12)$$

In Eq. (9) the prime indicates the derivative of the Legendre polynomial P with respect to its argument $(\hat{p}\cdot\hat{r})$.

These functions may now be directly compared with the corresponding ones for the Dirac-Coulomb case[6] for which all three functions are given as infinite series. The main difference is that the scattering solution for the Biedenharn Hamiltonian has simpler parameters in the ${}_1F_1$ functions and the function M is expressible in closed form. In fact, M is exactly the same as that for the Sommerfeld-Maue approximation, while N and L differ from the Sommerfeld-Maue approximation by order λ^2.

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Theory of Unstable Particles

E. Boldt
Department of Physics, Rutgers, The State University, New Brunswick, New Jersey

AND

G. Sandri
Aeronautical Research Associates of Princeton, Inc., Princeton, New Jersey

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A new method in the theory of unstable particles is introduced. It is applied in this paper to a simple model to show how the exponential regime of the decay can be isolated systematically. It is further shown that our method prescribes the precise conditions to which the initial wave function must be submitted for this exponential decay to ensue. The prescription of these conditions constitutes in fact a definition of an "unstable particle" in quantum theory.

INTRODUCTION

In this paper we shall be concerned with the classification of the decay regimes of unstable particles. The first task of the theory is that of isolating the exponential decay. It will be shown, by means of an illustrative model, how this can be accomplished systematically. Furthermore, it will be shown that the method is sufficiently powerful to allow for the determination of the precise conditions to be imposed on the initial wave packet for such an exponential decay.

The mathematical apparatus has been introduced recently by one of us[1] and applied to nonequilibrium statistical mechanics. Analogies with the results of Ref. 1 are numerous and will occasionally be noted. Previous treatments of unstable states are surveyed in Ref. 2.

For illustration we shall carry out our calculations with a model due to Wigner and Weisskopf.[3,4]

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